جامعة بنها – كلية العلوم - الفصل الدراسي الأول للعام 2019/2018

امتحان المستوى الرابع – شعبة رياضيات

المادة / تحليل دالى 411 ر | الزمن / ساعتان

Answer the following questions :

Question(1):

(a) Let $p>1, q>1: \frac{1}{p}+\frac{1}{q}=1$. show that if $x \in l^p$ and $y \in l^q$, then $xy \in l^1$. (7 marks)

(b) Show that $l^p \subseteq C_\circ \subseteq C \subseteq l^\infty$ and $U_{p\geq 1}\, l^p \subset_{\neq} l^\infty$ (7 marks)

(c) Consider the discrete metric space $(X, d_\circ)$ where $d_\circ(x,y)=\begin{cases} 0 & , \quad x=y \\ 1 & , \quad x\neq y \end{cases}$

Find $B(x_\circ 1)$, $\overline{B}(x_\circ 1)$, $S(x_\circ 2)$ (6 marks)

Question(2):

(a) Prove that $(l^p , \|\,.\,\|_p)$ is a B-space . (7 marks)

(b) State and prove the Banach contraction mapping theorem . (6 marks)

(c) Consider the two norms $\|x\|_1 = (|x_1|+|x_2|+|x_3|), \|x\|_2 = \left(|x_1|^2+|x_2|^2+|x_3|^2\right)^{1/2}$ Show that

$$\frac{1}{\sqrt{3}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \ , \ x=(x_1,x_2,x_3)\in R^3$$ (7 marks)

Question(3):

(a) Prove that a linear operator $T: X \to Y$ from a normed space X into a normed space Y is continuous iff it is bounded . (7 marks)

(b) Let $T: l^2 \to l^2 : T(x_1,x_2,\ldots,x_n,x_{n+1},\ldots)=(0,0,\ldots,0,x_{n+1},\ldots)$.

Show that T is continuous (6 marks)

(c) Let A and B be two convex sets . Show that $A+B=\{a+b : a\in A , b\in B\}$ is also a convex set .

(7 marks)

Question(4):

(a) *Let* $u=(x_1,x_2)$ *and* $v=(y_1,y_2)$. *Show that* $\langle u,v\rangle = x_1y_1+2x_2y_2$ *defines an inner product function . Find the angle between the two vectors* $(1,2)$ *and* $(2,3)$. *(7 marks)*

(b) *Let* H *be a Hilbert space and* $L\subset H$. *Prove that* L^1 *is a closed subspace of* H. *(7 marks)*

(c) *Let* (x_n) *be a sequence in a normed space* $(X,\|.\|)$. *Show that*

(i) *If* $x_n \overset{W}{\rightarrow} x_\circ$ *, then* $x_{n_k} \overset{W}{\rightarrow} x_\circ$.

(ii) *If* $x_n \overset{S}{\rightarrow} x_\circ$ *, then* $x_n \overset{W}{\rightarrow} x_\circ$. *(6 marks)*

مع أطيب التمنيات بالنجاح

إجابة اختبار مادة تحليل دالى 411 ر المستوى الرابع شعبة رياضيات - كلية العلوم العام الدراسي 2019/2018
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Question(1):

$(a)\ x(x_1,x_2,\ldots)\in l^p \Rightarrow \sum_{i=1}^{\infty}|x_i|^p<\infty$ *and* $y(y_1,y_2,\ldots)\in L^q \Rightarrow \sum_{i=1}^{\infty}|y_i|^q<\infty$

From Holder's Inequality we have

$$\sum_{i=1}^{\infty}|x_i y_i| \le \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p}\cdot\left(\sum_{i=1}^{\infty}|y_i|^q\right)^{1/q}<\infty \Rightarrow xy\in l^1$$

$(b)\ x(x_1,x_2,\ldots)\in L^p \Rightarrow \sum_{i=1}^{\infty}|x_i|^p<\infty \Rightarrow \lim_{i\to\infty}|x_i|^p=0 \Rightarrow \lim_{i\to\infty}|x_i|=0 \Rightarrow x\in C_\circ, C_\circ\subseteq C$

But since every conv.seqe. is bounded it follows that $\subseteq l^\infty$, $\therefore\ l^p\subseteq C_\circ\subseteq C\subseteq l^\infty$.

To prove that $l^p \subset_{\neq} l^\infty\ \forall\, p\ge 1$, *we have to consider the bounded seqe.* $\mathbf{(1,1,1,\ldots)\in l^\infty}$ *but*

$\sum_{n=0}^{\infty}(1)^n=\infty \Rightarrow (1,1,1,\ldots)\notin l^\infty\ \forall\, p\ge 1 \Rightarrow U_{p\ge 1}\, l^p \subset_{\neq} l^\infty.$

$$(c)\ B(x_\circ,1)=\{y\in X: d_\circ(x_\circ,y)<1\}=\{x_\circ\}$$
$$\overline{B}(x_\circ,1)=\{y\in X: d_\circ(x_\circ,y)\le 1\}=X$$
$$C(x_\circ,2)=\{y\in X: d_\circ(x_\circ,y)=2\}=\phi$$

Question(2):

(a) *To prove that* $(l^p,\|.\|_p)$ *is a B-space we have to prove that* (l^p, d_p) *is a complete metric space and* $\|.\|_p$ *is a norm on* l^p, *where* $d_p(x,y)=\left(|x_i-y_i|^p\right)^{1/p}$.

(b) *Banach contraction mapping theorem states that :*

"Every contraction mapping on a complete metric space has one and only one fixed point "

$(c)\ \|x\|_2=\left(|x_1|^2+|x_2|^2+|x_3|^3\right)^{1/2}\le\left(|x_1|+|x_2|+|x_3|\right)=\|x\|_1 \qquad (1)$

$\left(|x_1|+|x_2|+|x_3|\right)^2=|x_1|^2+|x_2|^2+|x_3|^2+2|x_1||x_2|+2|x_1||x_3|+2|x_2||x_3|\le(3)$

$\therefore \|x\|_1^2\le 3\left(|x_1|^2+|x_2|^2+|x_3|^2\right)\Rightarrow \frac{1}{\sqrt{3}}\|x\|_1\le\left(|x_1|^2+|x_2|^2+|x_3|^2\right)^{1/2}=\|x\|_2 \qquad (2)$

from $(1),(2)\ \frac{1}{\sqrt{3}}\|x\|_1\le\|x\|_2\le\|x\|_1$

Question(3):

(a) *Theorem*

$(b)\ T: l^2 \to l^2 : T(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (0,0,\ldots,0,x_{n+1},\ldots) \Rightarrow \| T(x) \|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} = \| x \|_2 \Rightarrow T$

is bounded and it is easy to show that T *is linear . i.e.* T *is continuous .*

(c) *We have to prove that for any* $(a_1 + b_1)$ *and* $(a_2 + b_2)$ *in* $A + B$,

$\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$ *(1)*

But the L.H.S of (1) is

$(\lambda a_1 + (1-\lambda) a_2) + (\lambda b_1 + (1-\lambda) b_2) \in A + B$

$(1)_A$ $(1)_B$

as both A *and* B *are convex .*

Question(4):

(a) *We have to prove the i.p.* $\langle u, v \rangle = x_1 y_1 + 2 x_2 y_2$ *satisfies the axioms of the i.p.*

$(b)\ L^1 = \{ y \in H : \langle y, x \rangle = 0\ \forall\ x \in L \}$. *Let* $x, y \in L^1 \Rightarrow \langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle = 0 + 0 = 0\ \ \forall\ z \in L$

$\therefore \lambda x + y \in L^1 \Rightarrow L^1 \leq H$. *To prove that* L^1 *is closed , we have to prove that* $\overline{L^1} \subseteq L^1$.

Let $x_\circ \in \overline{L^1} \Rightarrow \exists$ *a seq.* (x_n) *in* $L^1 : x_n \to x_\circ \Rightarrow \langle x_\circ, z \rangle = \langle \lim_{n\to\infty} x_n, z \rangle = \lim_{n\to\infty} \langle x_n, z \rangle = 0$ *where*

$z \in H \Rightarrow x_\circ \in L^1 \Rightarrow L^1$ *is closed .*

$(c)\ (i)\ x_n \xrightarrow{W} x_\circ \Rightarrow f(x_n) \to f(x_\circ)$, $f \in X'$, *but since* $(f(x_n))$ *is a sequence of real numbers , it*

follows that $f(x_{n_k}) \to f(x_\circ) \Rightarrow x_{n_k} \xrightarrow{W} x_\circ$

$(ii)\ x_n \xrightarrow{S} x_\circ \Rightarrow \| x_n - x_\circ \| \to 0$ *as* $n \to \infty$ *but since* $| f(x_n) \to f(x_\circ) | \leq \| f \| \| x_n - x_\circ \| \to 0$ *as* $n \to \infty$