كلية العلوم
نموذج إجابة مادة: الفيزياء الاحصائية المتقدمة
لطلبة تمهيدي دكتوراة فيزياء نظرية
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Q1. Prove the following relation for the occupation number n_i due to Fermi-Dirac statistics, $n_i = \dfrac{g_i}{e^{(\varepsilon_i-\varepsilon_F)/KT}+1}$.

------------------------------- **Solution** -----------------------------------

Let the number of allowed states associated with the energy ε_i be g_i. Let us first calculate the number of ways of putting n_1 particles of N particles in one box, then n_2 out of $N-n_1$ in second, and so on until we have exhausted all of the particles. The number of ways of choosing n_1 particles out of N particles is given by

$$W = \prod_i \frac{g_i!}{n_i!(g_i-n_i)!} \qquad (1)$$

(٣)

$$\ln W = \sum_i [(\ln g_i! - \ln n_i! - \ln(g_i-n_i)!]$$

To obtain the most probable distribution, we maximize Eq. (3) with $dN = 0$:

$$\delta \ln W = \sum_i \ln \frac{g_i-n_i}{n_i}\delta n_i = 0$$

but

$$\delta N = \sum_i \delta n_i = 0 \qquad (4)$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \qquad (5)$$

multiply Eq. (4) by $-\alpha$ and Eq. (5) bt $-B$ and add the resulting equations to each other:

$$\sum_i \left[\ln\left(\frac{g_i - n_i}{n_i}\right) - \alpha - \beta\varepsilon_i \right] \delta n_i = 0 \qquad (6)$$

Since n_i is vary independent,

$$\ln\left(\frac{g_i - n_i}{n_i}\right) - \alpha - \beta\varepsilon_i = 0$$

Solving Eq. (7) for n_i gives

$$n_i = \frac{g_i}{e^{(\alpha+\beta\varepsilon_i)} + 1}$$

Q2. Define the term paramagnetic, then derive an expression, in terms of the partition function Z, for the internal energy, U, stored in a paramagnetic crystal as a result of an external applied magnetic field.

---------------------------------- Solution --------------------------------------

A paramagnetic substance in the absence of an external magnetic field is not a magnet . upon being introduced into a magnetic field it becomes slightly magnetized in the direction of the field .Certain paramagnetic crystals play an interesting and important role in modern physics ,particularly at very low temperatures. A typical paramagnetic crystaa is chromium potassium sulfate Cr2(So4)3 .K2SO4 .24H2o. Its Paramagnetic Properties are due to the chromium atoms , which exist in the crystal Aions Cr^{+++} . Consider a paramagnetic crystal consisting of N non-Interacting atoms and placed in an external magnetic field H pointing along the Z direction , see fig(1).

The magnetic energy of an atom can be written as :

$$\varepsilon_{m} = -\mu . H \qquad (1)$$

where μ is the magnetic moment of the atom . it is proportional to the total angular momentum

hJ of the atom and is conventional written is the form :

$$\mu = g \mu_B \mathbf{J} \qquad \mathbf{(2)}$$

where μ_B **is astandard unit of magnatic moment (usually the Bohr magneton,** $\mu_{B} = e\hbar/(2mc)$, m being electron mass) and g a number known as g – factor or splitting factor .

by combining Eqs(1)and (2) one obtains :

$$\varepsilon_{m} = - g \mu_B \mathbf{J} . \mathbf{H} = - g \mu_B H J_z \ , \qquad (3)$$

since H points in the z direction. in a quantum mechanical description the values which J_z Can assume are discrete and are given by **:**

$$J_z = m$$

where m take all values between – J and +J in integral steps :

$$m = -J , -J+1 , -J+2 ,…………, J-1 , J \qquad (4)$$

thus there are 2J+1 possible values of m corresponding to that many possible projections of the angular momentum vector along z axis . by virtue of Eq (3) , the possible magnetic energies of the are then :

$$\varepsilon_{\mathbf{m}} = - g\, \mu_B\, Hm \qquad (5)$$

The total energy of aparmagnetic atom , equal to the sum of the potential energy u_i due to the crystalline field and the magnetic energy $\varepsilon_{\mathbf{m}}$

due to the presence of an external magnetic field H . Thus :

$$\varepsilon_{\mathbf{i}} = u_i + \varepsilon_{\mathbf{m}} \qquad \mathbf{(6)}$$

the total energy due to n_i atoms is :

$$\sum n_i\, \varepsilon_{\mathbf{i}} = \sum n_i\ u_i + \sum n_i\, \varepsilon_{\mathbf{m}} \quad ,$$

$$\sum n_i\, \varepsilon_{\mathbf{i}} = \mathbf{U\text{-}H} \sum g\, \mu_B\, n_i\, m \qquad (7)$$

The sum $\sum g\, \mu_B\, n_i\, m$ is the magnetization M ,so

$$\sum n_i\, \varepsilon_{\mathbf{i}} = \mathbf{U\text{-}MH} \qquad \mathbf{(8)}$$

Q3. Describe, with the help of a diagram, the magnetic cooling by adiabatic demagnetization process

---------------------------------- **Solution** --------------------------------------

Refrigeration is the process of removing heat from matter which may be a solid, a liquid, or a gas. Removing heat from the matter cools it, or lowers its temperature. In the mechanical refrigeration a refrigerant is a substance capable of transferring heat that it absorbs at low temperatures and pressures to a condensing medium. In the region of transfer, the refrigerant is at higher temperatures and pressures. By means of expansion, compression, and a cooling medium, such as air or water, the refrigerant removes heat from a substance and transfers it to the cooling medium.

It is well known that the efficiency of the conventional gas compression expansion refrigeration system cannot significantly be improved. Also, for the conventional refrigeration system, there exist serious concerns for the environment. Thus, it is necessary and important to explore other alternative cooling technologies. The magnetic refrigerator, which has advantages in refrigeration efficiency, reliability, low noise and environmental friendliness with respect to the conventional gas refrigerators, is becoming a promising technology to replace the conventional technique.

The first ‘‘room temperature magnetic refrigerator’’ containing permanent magnets was designed and built in 2001 in the USA by the Astronautics Corporation. The early prototypes were able to reach high magnetic flux densities in the magneto caloric material only if superconducting magnets were applied.

The magneto caloric effect can be quantified with the equation below:

$$\Delta T_{ad} = -\int_{H_0}^{H_1} \left(\frac{T}{C(T,H)}\right)_H \left(\frac{\partial M(T,H)}{\partial T}\right)_H dH \quad , \qquad (1)$$

where T is the temperature, H is the applied magnetic field, C is the heat capacity of the working magnet (refrigerant), and M is the magnetization of the refrigerant. From the equation we can see that magneto caloric effect can be enhanced by applying a large field.

Let us calculate the entropy of paramagnetic crystal at low temperatures using Debye theory as

$$C_v = \frac{12\pi^4}{5} NK \left(\frac{T}{\theta_D}\right)^3 \qquad (1)$$

Where C_v is specific heat and Θ_D is Debye temperature . then entropy due to lattice is

$$S_L = \int_0^T \frac{C_V}{T} dT = \frac{4\pi^5}{5} NK \left(\frac{T}{\theta_D}\right)^3 \qquad (2)$$

So S_L is negligible compared to S_H "the entropy due to the magnetic field" which can be calculated from the magnetic partition function as

$$S_H = NK \left(\frac{\partial}{\partial T} T\, lnZ_H\right)_H$$

$$S_H = NKT \left(\frac{\partial lnZ_H}{\partial T}\right)_H + NK lnZ_H \qquad (3)$$

Since $Z_H = \frac{\sinh\left(J+\frac{1}{2}\right)x}{\sinh\frac{1}{2}x}$ with $x = \frac{g\mu_B H}{KT}$

So $\left(\frac{\partial lnZ_H}{\partial T}\right)_H = -\frac{x}{T} J B_J(x)$

So
$$\frac{S_H}{NK} = -JxB_J(x) + ln\frac{\sinh\left(J+\frac{1}{2}\right)x}{\sinh\frac{1}{2}x} \qquad (4)$$

at low temperature (x>>1),$B_J(x)$=1 , so

$$S_H = S_H(x) = S_H(H/T) \qquad (5)$$

at high temperature (x<<1), $\sinh a = a + \frac{1}{6}a^3$ *and* $\ln(1+x) \cong 1$

then

$$ln\frac{\sinh\left(J+\frac{1}{2}\right)x}{\frac{1}{2}x} = \ln(2J+1) + \frac{1}{2}JxB_J(x)$$

using this relation into eq.(4) gives

$$\frac{S_H}{NK} = -JxB_J(x) + \ln(2J+1) + \frac{1}{2}JxB_J(x)$$

$$\frac{S_H}{NK} = \ln(2J+1) - \frac{1}{2}JxB_J(x) \qquad (6)$$

but $B_J(x) = \frac{J+1}{3}x \; and \; x = \frac{g\mu_B H}{KT}$

so $\quad \frac{S_H}{NK} = \ln(2J+1) - \frac{1}{2}Jx.\frac{J+1}{3}x$

$$\frac{S_H}{NK} = \ln(2J+1) - \frac{J(j+1)}{6}x^2$$

$$\frac{S_H}{NK} = \ln(2J+1) - \frac{J(j+1)}{6}\left(\frac{g\mu_B H}{KT}\right)^2$$

$$\frac{S_H}{NK} = \ln(2J+1) - \frac{g^2{\mu_B}^2 J(J+1)}{6K^2}\left(\frac{H}{T}\right)^2 \qquad (7)$$

Eq. (7) shows the effect of an external magnetic field is to align the magnetic moments and introduce more order into the system causing a reduction in entropy as in fig(1).

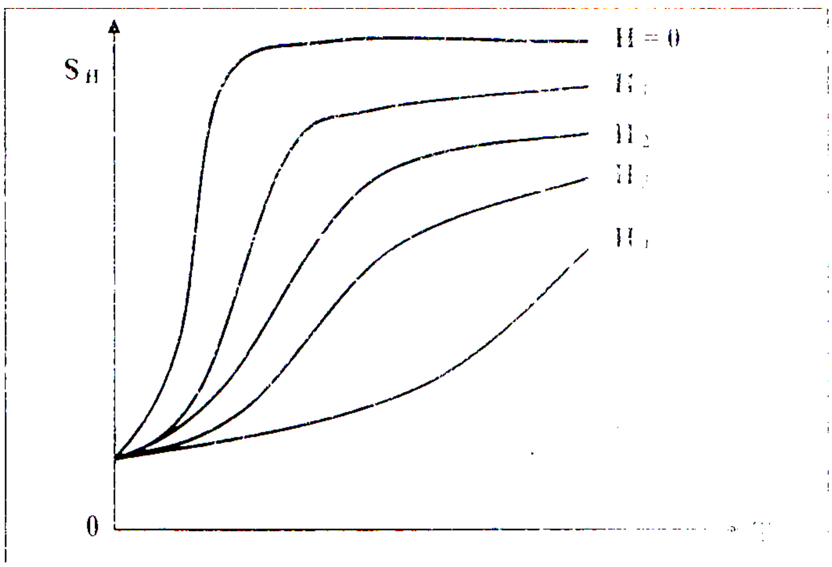

Fig. (1): Magnetic entropy versus temperature

We would define the Isothermal process and Adiabatic process, before explain how we can cooling the crystal.

Isothermal process: in this process the temperature of the substance is constant with the surrounding medium so the internal energy of the substance is constant ($\Delta U=0$).Then any acquired energy will convert to kinetic energy for dipole.

Adiabatic process: in which the substance isolated from the surround medium so the substance does not acquire or loss heat. The movement(work) of dipole depend on the internal energy of substance and happen the following:

1. If the dipole done work the internal energy decreases and so the temperature decreases.
2. If we done work on the dipole the internal energy increases and the temperature increases.

We can use the previous processes and magnetic field for cooling substances this method which consists of two steps:

1. The isothermal magnetization of the crystal which is thermal contact with liquid helium.
2. The crystal is then insulated and adiabatically demagnetization causing decrease in temperature.

The result of these steps are plotted in fig2. In the isothermal increase of the magnetic field a→b the entropy decreases. In the adiabatic decrease of the magnetic field b→c the temperature decreases.

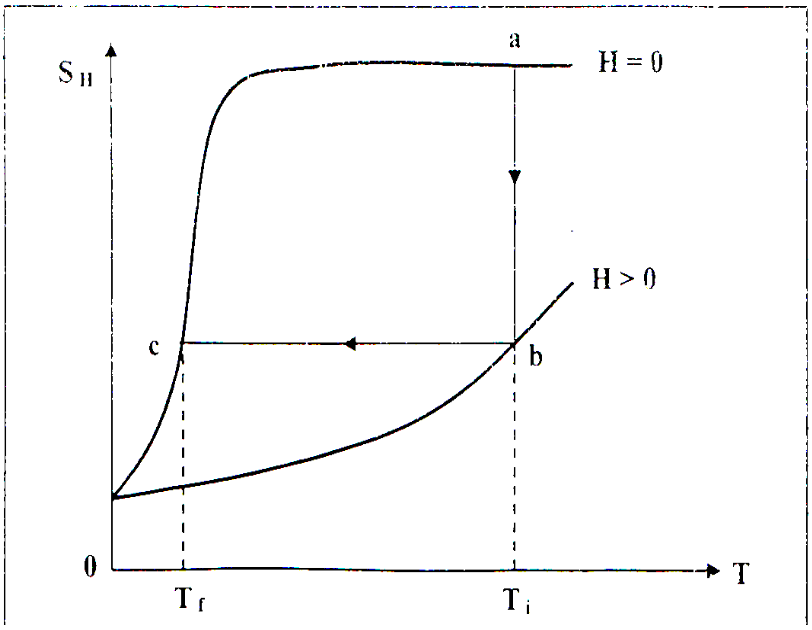

Fig. (2): Magnetic entropy versus temperature

The cycle is performed as a refrigeration cycle where H is the externally applied magnetic field, Q is the heat quantity and P is the pressure. Analogous to the Carnot cycle, The thermodynamic cycle can be described, as in Fig. (3), at a starting point whereby the chosen working substance is introduced into a magnetic field, i.e., the magnetic flux density is increased. The working material is the refrigerant, and starts in thermal equilibrium with the refrigerated environment.

- ***An adiabatic magnetization***: a magneto caloric substance is placed in an insulated environment. The increasing external magnetic field (+H) causes the magnetic dipoles of the atoms to align, thereby decreasing the material's magnetic entropy and heat capacity. Since overall energy is not lost (yet) and therefore total entropy is not reduced (according to thermodynamic laws), the net result is that the item heats up ($T + \Delta T_{ad}$).

- ***An adiabatic Demagnetization***: refrigerator (ADR) works by using the properties of heat and the magnetic properties of certain molecules. Some molecules have large internal magnetic fields, or "moments". Just like a tiny bar magnet, these molecules will align themselves with an external magnetic field. The random thermal motions of the molecules, on the other hand, tend to de-align them. The higher the temperature, the more they de-align.

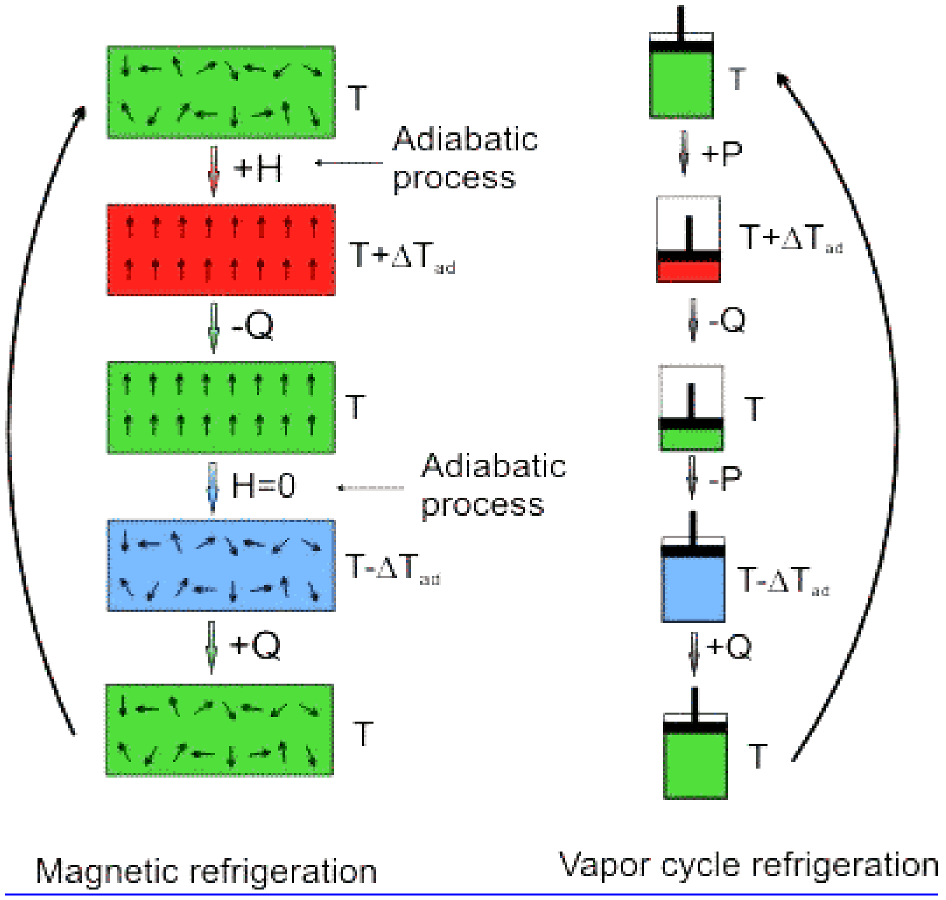

Fig. (3): Thermodynamic Cycle

Q4. **Describe, with the help of a diagram, the magnetic cooling by adiabatic demagnetization process.**

-------------------------------- Solution ------------------------------------

Magnetic cooling:

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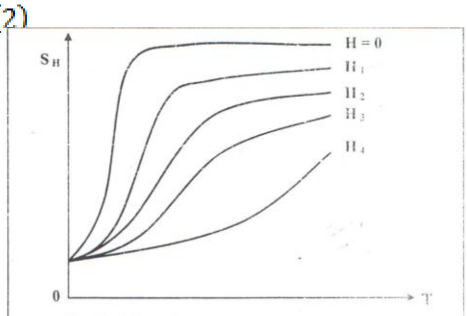

Fig. (1): Magnetic entropy versus temperature

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$$So \qquad \frac{S_H}{NK} = -JxB_J(x) + ln\frac{\sinh\left(J+\frac{1}{2}\right)x}{\sinh\frac{1}{2}x} \qquad (4)$$

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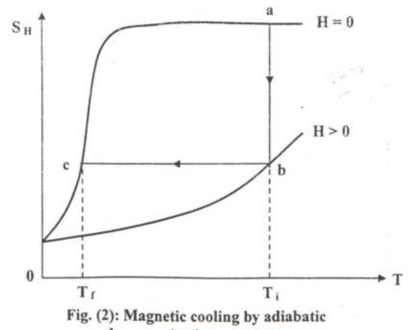

Fig. (2): Magnetic cooling by adiabatic demagnetization processes.

The result of these steps are plotted in fig2. In the isothermal increase of the magnetic field a→b the entropy decreases. In the adiabatic decrease of the magnetic field b→c the temperature decreases.