جامعة بنها　　　　　　　　　　　　　　الفرقة الثالثة (فيزياء)
كلية العلوم　　　　　　　　　　　　　　مادة (إحصائية)
دور يناير ٢٠١٤　　　تخلف نظام قديم　　　الزمن ٣ ساعات

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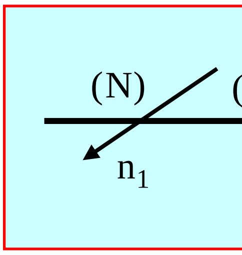
1. Prove the following relation for the occupation number n_i due to Boltzmann distribution $n_i = \sum_i \frac{N}{Z} e^{\varepsilon_i / KT}$

------------------------------- **Solution** ---------------------------------

Let the number of allowed states associated with the energy ε_i be g_i. Let us first calculate the number of ways of putting n_1 particles of N particles in one box, then n_2 out of $N - n_1$ in second, and so on until we have exhausted all of the particles. The number of ways of choosing n_1 particles out of N particles is given by



$$W_1 = \frac{N!}{(N - n_1)!\, n_1!} \tag{1}$$

and the number of choosing n_2 out of $N - n_1$ is:

$$W_2 = \frac{(N - n_1)!}{(N - n_1 - n_2)!\, n_2!} \qquad (٢)$$

and the number of ways of achieving this arrangement is

$$W = W_1 \cdot W_2 \ \cdots$$

$$= \frac{N!}{(N-n_1)!\ n_1!} \cdot \frac{(N-n_1)!}{(N-n_1-n_2)!\ n_2!} \ \cdots$$

$$= \frac{N!}{n_1!\ n_2! \ \cdots \ n_i!}$$

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i} \qquad (٣)$$

$$\ln W = \ln N! + \sum_i (n \ln g_i - n \ln n_i!)$$
$$= N \ln N + \sum_i (n \ln g_i - n \ln n_i)$$

To obtain the most probable distribution, we maximize Eq. (3) with

$dN = 0$:

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - \frac{n_i}{n_i}) \delta n_i = 0$$

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - 1) \delta n_i = 0$$

but

$$\delta N = \sum_i \delta n_i = 0 \qquad (4)$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \qquad (5)$$

multiply Eq. (4) by $\alpha + 1$ and Eq. (5) bt $-B$ and add the resulting

equations to each other:

$$\sum_i (\ln g_i - n \ln n_i + \alpha - \beta\varepsilon_i)\delta n_i = 0 \qquad (6)$$

Since n_i is vary independent,

$$\ln g_i - n \ln n_i + \alpha - \beta\varepsilon_i = 0$$

or

$$\ln \frac{g_i}{n_i} + \alpha - \beta\varepsilon_i = 0 \qquad (7)$$

Solving Eq. (7) for n_i gives

$$n_i = \frac{N}{Z} g_i e^{-\beta\varepsilon_i}$$

2. (a) Find the relation between the partition function Z and thermodynamic functions U, S, and F.

-------------------------------- Solution --------------------------------

(a) Relation between Z and U

Since

$$Z = \sum_i g_i e^{\varepsilon_i / KT}$$

differentiate Z with respect to T, holding V constant,

$$\left(\frac{\partial Z}{\partial T}\right)_V = \sum_i g_i \left(\frac{\varepsilon_i}{KT^2}\right) e^{\varepsilon_i / KT}$$

$$= \frac{1}{KT^2} \sum_i \varepsilon_i g_i \; e^{\varepsilon_i / KT}$$

$$= \frac{1}{KT^2} \frac{\sum_i n_i \varepsilon_i}{\sum_i n_i} g_i \; e^{\varepsilon_i / KT}$$

$$= \frac{ZU}{NKT^2}$$

It follow that

$$U = NKT^2 \left(\frac{\partial \ln Z}{\partial T}\right)_V \qquad (8)$$

and U may be calculated once lnZ is known as a function of T and V.

(b) Relation between Z and S

The entropy S is related to the order or distribution of the particles, through the relation:

$$S = K \ln W$$

but

$$\ln W = -\sum_i n_i \ln \frac{n_i}{g_i} + N \ln N$$

Hence

$$S = K \ln W = K\left[-\sum_i n_i \ln \frac{n_i}{g_i} + N \ln N\right]$$

By using the relation

$$n_i = \frac{N}{Z} g_i e^{-\varepsilon_i / KT}$$

we have

$$\frac{n_i}{g_i} = \frac{N}{Z} e^{-\varepsilon_i / KT}$$

then

$$\begin{aligned} S = K \ln W &= K\left[-N \ln N + N \ln Z + \frac{U}{KT} + N \ln N \right] \\ &= NKT \ln Z + \frac{U}{T} \end{aligned} \qquad (9)$$

and S may be calculated once lnZ is known as a function of T and V.

(c) Relation between Z and F

The property of the system is defined by its Helmholtz function F which is given by:

$$F = U - TS$$

This equation can be evaluated in terms of the partition function Z. By using the entropy S, Eq. (8), we get

$$F = U - T\left(NK \ln Z + \frac{U}{T} \right)$$

$$F = -NKT \ln Z \qquad (10)$$

and F may be calculated once lnZ is known as a function of T and V.

2. Prove the following relation for the occupation number n_i due to Fermi-Dirac statistics, $n_i = \dfrac{g_i}{e^{(\varepsilon_i - \varepsilon_F)/KT} + 1}$.

---------------------------------- **Solution** --------------------------------------

Let the number of allowed states associated with the energy ε_i be g_i. Let us first calculate the number of ways of putting n_1 particles of N particles in one box, then n_2 out of $N - n_1$ in second, and so on until we have exhausted all of the particles. The number of ways of choosing n_1 particles out of N particles is given by

$$W = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!} \qquad (1)$$

(٣)

$$\ln W = \sum_i [(\ln g_i! - \ln n_i! - \ln (g_i - n_i)!]$$

To obtain the most probable distribution, we maximize Eq. (3) with $dN = 0$:

$$\delta \ln W = \sum_i \ln \frac{g_i - n_i}{n_i} \delta n_i = 0$$

but

$$\delta N = \sum_i \delta n_i = 0 \qquad (4)$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \qquad (5)$$

multiply Eq. (4) by $-\alpha$ and Eq. (5) bt $-B$ and add the resulting

equations to each other:

$$\sum_i \left[\ln\left(\frac{g_i - n_i}{n_i} \right) - \alpha - \beta\varepsilon_i \right] \delta n_i = 0 \qquad (6)$$

Since n_i is vary independent,

$$\ln\left(\frac{g_i - n_i}{n_i} \right) - \alpha - \beta\varepsilon_i = 0$$

Solving Eq. (7) for n_i gives

$$n_i = \frac{g_i}{e^{(\alpha + \beta\varepsilon_i)} + 1}$$