Benha University **Department of Mathematics**

Faculty of science **Faculty of Science** **Fourth year -**
(topology)

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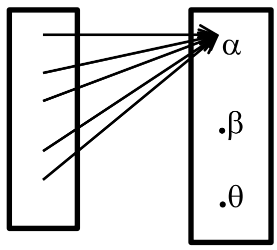
Model answer:

(1) The function h is continuous because



$h^{-1}(\mathrm{Y}) = \mathrm{X} \in \tau$, $h^{-1}(\phi) = \phi \in \tau$,

$h^{-1}(\{\alpha\}) = \mathrm{X} \in \tau$ and

$h^{-1}(\{\alpha, \theta\}) = \{\mathrm{X}, \phi\} \in \tau$.

Also, the function h is open because for each open set G of (X, τ), we have

$$h(G) = \begin{cases} \{\alpha\}, & \text{if } G \neq \phi \\ \phi, & \text{if } G = \phi. \end{cases}$$

Which are open sets of (Y, σ)

The function h is not closed because for each closed set $F \neq \phi$ of (X, τ) we have $h(F) = \{\alpha\}$ is not closed of (Y, σ).

(2)

(m1) $d(\mathrm{x}, \mathrm{y}) = |\mathrm{x} - \mathrm{y}| > 0$ and $d(\mathrm{x}, \mathrm{x}) = |\mathrm{x} - \mathrm{x}| = 0$

(m2) $d(\mathrm{x}, \mathrm{y}) = |\mathrm{x} - \mathrm{y}| = |\mathrm{y} - \mathrm{x}| = d(\mathrm{y}, \mathrm{x})$

(m3) $d(\mathrm{x}, \mathrm{z}) = |\mathrm{x} - \mathrm{z}| = |\mathrm{x} - \mathrm{y} + \mathrm{y} - \mathrm{z}| \leq |\mathrm{x} - \mathrm{y}| + |\mathrm{y} - \mathrm{z}|$

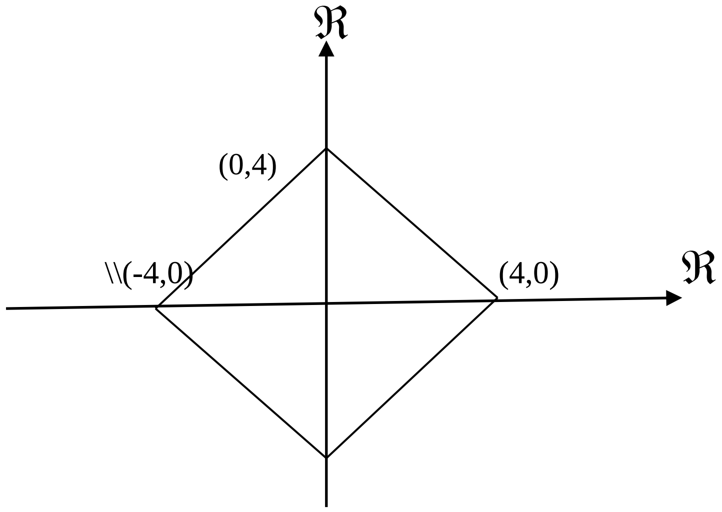
$= d(\mathrm{x}, \mathrm{y}) + d(\mathrm{y}, \mathrm{z})$.

$$\mathrm{S}(\mathrm{p}, 4) = \{\mathrm{q} \equiv (\mathrm{x}, \mathrm{y}) \in \Re^2 \quad : \quad d(\mathrm{q}, \mathrm{p}) < 4\}$$

$$=\{(x,y)\in\Re^2 \quad : \quad |x-0|+|y-0|<4\}$$

$$=\{(x,y)\in\Re^2 \quad : \quad |x|+|y|<4\}$$

Will be the subset of $\Re^2$ which cuts the oX axis at (-4,0) , (4,0) cuts the and cuts the oY axis at (0,-4) , (0,4) . Illustrated in the figure.



(3)

Since $q \in S(p, \delta)$, then $d(p, q) < \delta$. Hence

$\varepsilon = \delta - d(p, q) > 0$. We take $T = T(q, \varepsilon)$.

To prove that $T \subseteq S$, let $x \in T(q, \varepsilon)$. Then

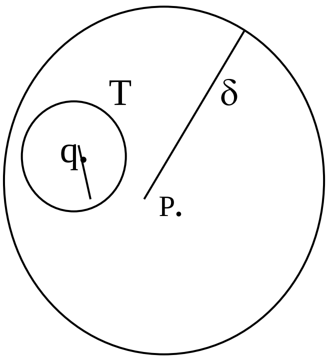
$d(x, q) < \varepsilon = \delta - d(p, q)$.

So, by triangle inequality, we have

$$d(x, p) \leq d(x, q) + d(q, p)$$
$$< [\delta - d(p, q)] + d(q, p)$$
$$= \delta.$$

Thus, $x \in S(p, \delta)$.

Which proves $T \subseteq S$.



(4)

Suppose that $p \in \{x \in X \quad : \quad d(x, A) = 0\}$.

So that, $d(\mathrm{p}, A) = 0$. Which means that every open sphere with center p will contains at least one point of A. i. e.

$$\mathrm{S}(\mathrm{p}, \delta) \cap A \neq \phi \quad \forall \ \delta > 0$$

$$\Rightarrow \ V \cap A \neq \phi \quad \forall \quad V = \mathrm{S}(\mathrm{p}, \delta) \in \mathcal{N}_{\mathrm{p}}$$

$$\Rightarrow \ (V - \{\mathrm{p}\}) \cap A \neq \phi \quad \forall \quad V \in \mathcal{N}_{\mathrm{p}}$$

$$\Rightarrow \ \mathrm{p} \in A^{/}$$

$$\Rightarrow \ \mathrm{p} \in A \cup A^{/}$$

$$\Rightarrow \ \mathrm{p} \in \overline{A}$$

$$\Rightarrow \{\mathrm{x} \in \mathrm{X} \quad : \quad d(\mathrm{x}, A) = 0\} \subseteq \overline{A} \quad \ldots (1)$$

On the other hand, let $\mathrm{q} \notin \{\mathrm{x} \in \mathrm{X} \quad : \quad d(\mathrm{x}, A) = 0\}$

$$\Rightarrow \ d(\mathrm{q}, A) \neq 0$$

$$\Rightarrow \ d(\mathrm{q}, A) = \delta \quad \text{for some} \ \ \delta > 0$$

Thus, there are two facts, the first one is $\mathrm{q} \notin A$.

The second fact is that, the open sphere $\mathrm{S}(\mathrm{p}, ½ \, \delta)$ with center p and radius $½\delta$ not contains any point of A. i.e.

$$\mathrm{S}(\mathrm{q}, ½\delta) \cap A = \phi$$

$$\Rightarrow \ U \cap A = \phi, \quad \text{for some} \ \ U = \mathrm{S}(\mathrm{q}, ½\delta) \in \mathcal{N}_{\mathrm{q}}$$

$$\Rightarrow \ (U - \{\mathrm{q}\}) \cap A = \phi, \quad \text{for some} \ \ U \in \mathcal{N}_{\mathrm{q}}$$

$$\Rightarrow \ \mathrm{q} \notin A^{/}$$

$$\Rightarrow \mathrm{q} \notin A \cup A^{/} = \overline{A}$$

$$\Rightarrow \ \overline{A} \subseteq \{\mathrm{x} \in \mathrm{X} \quad : \quad d(\mathrm{x}, A) = 0\} \quad \ldots (2)$$

$$\overset{(1)}{\underset{(2)}{\Rightarrow}} \ \overline{A} = \{\mathrm{x} \in \mathrm{X} \quad : \quad d(\mathrm{x}, A) = 0\}$$

(5)

Let (X, τ) be a T_1-space, we must show that every singleton set is

τ-closed. That is, for each $\alpha \in X$, we show that $X-\{\alpha\}$ is a τ-open subset.

Now, let $\alpha \neq \beta$ in X, we have $\alpha \in X - \{\beta\}$ and $\beta \in X - \{\alpha\}$. Then, by Definition 5-2, there are a neighborhood U of α such that $\beta \notin U$ and a neighborhood V of β such that $\alpha \notin V$,

i.e. $\{\alpha\} \cap V = \phi$, that is $\beta \in V \subseteq X - \{\alpha\}$.

So, for every point β in X there is $G_\beta \in \tau$ such that $\beta \in G_\beta \subseteq V \subseteq X - \{\alpha\}$

that is $\bigcup_{\beta} G_\beta = X - \{\alpha\}$.

Which means that the subset $X - \{\alpha\}$ is a τ-open subset.

Conversely; consider that every singleton subset $\{\alpha\}$ of X is τ-closed and $\alpha \neq \beta$ in X. Then $X - \{\alpha\}$ is a τ-open, thus it is a neighborhood of β not containing α and $X - \{\beta\}$ is a τ-open, also it is a neighborhood of α not containing β.

Hence, (X, τ) is a T_1-space.