Benha University
Faculty of Science
Dept. of Mathematics

Time: 3 Hours
Second Term 2016-2017
Date: 21/6/2017

Complex Variables (412 M) *for Fourth Year Students,*
(Math. *Section*)

جامعة بنهـــا – كلية العلوم – قسم الرياضيات

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المادة : متغير مركب (تحليل مركب) (412 M)

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مدرس بقسم الرياضيات بكلية العلوم

الامتحان+ نموذج إجابة

ورقة كاملة

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<u>Answer the following questions</u>

<u>Question 1.</u> [12 mark]

(a) Obtain all values of $\log(1-i\sqrt{3})$ & $(1+i)^i$.

(b) Show that $\cos\bar{z}$ isn't analytic function of z anywhere.

<u>Question 2.</u> [20 mark]

(a) Use Cauchy's integrals to evaluate:

$$K=\int_C \frac{z^2 e^{\pi z}}{(z-i)^2}dz,\qquad L=\int_C \frac{z^2+e^{z^2}}{z^3}dz,\qquad M=\int_C \frac{z^2 e^{3z}}{(z-i\frac{\pi}{2})^2}dz.$$

where C is the circle $|z-i|=2$.

(b) If $w=u+iv$ is analytic, and $u=2x-x^3+3xy^2$ then find v and obtain w & $\dfrac{dw}{dz}$ in terms of z.

<u>Question 3.</u> [20 mark]

(a) Obtain expansions in powers of $(z+1)$ for the function $f(z)=\dfrac{z^3-z+1}{z-2}$ and give regions of validity.

(b) Find the Maclaurin series for the function $\log(1+z^2)$.

<u>Question 4.</u> [28 mark]

Use contour integration to evaluate the following integrals:

(i) $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$, (ii) $\displaystyle\int_0^\infty \frac{\cos x\,dx}{(x^2+1)^2}$,

(iii) $\displaystyle\int_{-\pi}^{\pi} \frac{\cos\theta\,d\theta}{5+4\cos\theta}$, (iv) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}$.

Good Luck!

د/ محمد السيد نصر

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The answer

Answer Question 1:

(a) Obtain all values of $\log(1-i\sqrt{3})$ & $(1+i)^i$.

$$\log(1-i\sqrt{3}) = \log\left[2e^{-i\frac{\pi}{3}}\right] = \ln 2 + i(2\pi k - \frac{\pi}{3}),\quad k = 0,1,2,3,\ldots.$$

$$(1+i)^i = \left[\sqrt{2}e^{i\frac{\pi}{4}}e^{2\pi ki}\right]^i = \left[e^{\ln\sqrt{2}}e^{i(\frac{\pi}{4}+2\pi k)}\right]^i = e^{-(\frac{\pi}{4}+2\pi k)}e^{i\ln\sqrt{2}}$$

$$= e^{-\frac{\pi}{4}(1+8k)}\left[\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})\right],$$

$$k = 0,1,2,3,\ldots.$$

(b) Show that $\cos\bar{z}$ isn't an analytic function of z anywhere.

$$\cos\bar{z} = \cos(x-iy) = \cos x\cos iy + \sin x\sin iy$$

$$\cos\bar{z} = \cos x\cosh y + i\sin x\sinh y = u + iv$$

$$\Rightarrow\ u = \cos x\cosh y \qquad \& \quad v = \sin x\sinh y$$

$$\Rightarrow\ \frac{\partial u}{\partial x} = -\sin x\cosh y \quad \& \quad \frac{\partial v}{\partial y} = \sin x\cosh y,$$

$$\frac{\partial u}{\partial y} = \cos x\sinh y \qquad \& \quad \frac{\partial v}{\partial x} = \cos x\sinh y$$

$$\Rightarrow\ \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial y}$$

CRC aren't satisfied

So, $\cos\bar{z}$ isn't an analytic function.

Answer Question 2:

(a)

$$K=\int_C \frac{z^2 e^{\pi z}}{(z-i)^2}dz = 2\pi i\left[\frac{d}{dz}z^2 e^{\pi z}\right]=2\pi i[(2z+\pi z^2)e^{\pi z}]_{z=i}$$

$$=2\pi i[(2i-\pi)e^{\pi i}]=2\pi i[(\pi-2i)]=2\pi(2+i\pi),$$

$$L=\int_C \frac{z^2+e^{z^2}}{z^3}dz=\frac{2\pi i}{2}\left[\frac{d^2}{dz^2}(z^2+e^{z^2})\right]_{z=0}=\pi i\left[\frac{d}{dz}(2z+2z\,e^{z^2})\right]_{z=0}$$

$$=\pi i(2+2e^{z^2}+4z^2e^{z^2})_{z=0}=\pi i(2+2)=4\pi i,$$

$$M=\int_C \frac{z^2 e^{3z}}{(z-i\frac{\pi}{2})^2}dz=2\pi i\left[\frac{d}{dz}z^2 e^{3z}\right]=2\pi i[(2z+3z^2)e^{3z}]_{z=i\frac{\pi}{2}}$$

$$=2\pi i\left[(\pi i-\frac{3\pi^2}{4})(-i)\right]=2\pi^2(i-\frac{3\pi}{4})=\frac{\pi^2}{2}(-3\pi+4i)$$

(b)

$$\because\ w \text{ is analytic} \Rightarrow \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Longrightarrow \Rightarrow v=(2-3x^2)y+y^3+\phi(x)$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\Rightarrow \frac{\partial v}{\partial x}=-6xy=\phi'(x)+(-6x)y\Rightarrow \phi'(x)=0 \Rightarrow \phi=c \Rightarrow$$

$$v=(2-3x^2)y+y^3+c\Rightarrow w=u+iv=2x-x^3+3x\,y^2+i((2-3x^2)y+y^3+c)$$

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2-3x^2+3y^2-i6xy=2-3(x^2-y^2+i2x\,y)=$$

$$=2-3(x+iy)^2=2-3z^2.$$

Answer Question 3:

(a)

$$f(z)=\frac{z^3-2z^2+2z^2-z+1}{z-2}=z^2+\frac{2z^2-4z+3z+1}{z-2}=z^2+2z+\frac{3z-6+7}{z-2}=$$

$$=z^2+2z+3+\frac{7}{z-2}=(z+1)^2+2+\frac{7}{z+1-3}$$

at $|z+1|<3 \Rightarrow \left|\frac{z+1}{3}\right|<1$

$$f(z)=\frac{z^3-z+1}{z-2}=(z+1)^2+2-\frac{7}{3}\frac{1}{1-\frac{z+1}{3}}=$$

$$=(z+1)^2+2-\frac{7}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

at $|z+1|>3 \Rightarrow \left|\frac{3}{z+1}\right|<1$

$$f(z)=\frac{z^3-z+1}{z-2}=(z+1)^2+2+\frac{7}{(z+1)}\frac{1}{1-\frac{3}{z+1}}=$$

$$=(z+1)^2+2+\frac{7}{(z+1)}\left[1+\frac{3}{z+1}+\left(\frac{3}{z+1}\right)^2+\cdots\right]$$

(b)

$$\frac{2\zeta}{1+\zeta^2} = 2\zeta(1-\zeta^2+\zeta^4-\zeta^6+\cdots) =$$

$$= 2(\zeta-\zeta^3+\zeta^5-\zeta^7+\cdots), \quad \left|\zeta^2\right|<1$$

integration both sides $\zeta : 0 \to z$

$$\log(1+\zeta^2)\Big|_0^z = 2\left\{\frac{\zeta^2}{2}-\frac{\zeta^4}{4}+\frac{\zeta^6}{6}-\frac{\zeta^8}{8}+\cdots\right\}\Bigg|_0^z$$

$$\log(1+z^2) = 2\left\{\frac{z^2}{2}-\frac{z^4}{4}+\frac{z^6}{6}-\frac{z^8}{8}+\cdots\right\}, \quad |z|<1$$

$$= \sum_{n=1}(-1)^{n+1}\frac{(z)^{2n}}{n}, \quad |z|<1$$

Answer Question 4:

(i)

$$I_1=\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)(x^2+4)}$$

$$f(z)=\frac{z^2}{(z^2+1)(z^2+4)}=\frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)} \quad \Rightarrow \quad z_0=\pm i,\ \pm 2i$$

$$b_1(i)=\left[\frac{z^2}{(z+i)(z-2i)(z+2i)}\right]_{z=i}=\frac{-1}{(2i)(3)}=\frac{i}{6}$$

$$b_1(2i)=\left[\frac{z^2}{(z+i)(z-i)(z+2i)}\right]_{z=2i}=\frac{-4}{(4i)(-3)}=\frac{-i}{3}$$

$$\int_C f(z)\,dz=2\pi i\left(\frac{i}{6}-\frac{i}{3}\right)=2\pi(-1)\left(-\frac{1}{6}\right)=\frac{\pi}{3}=2I_1+\int_{C'} f(z)\,dz \quad (as\ R\to\infty)$$

$$\text{but}\ \int_{C'} f(z)\,dz=\int_0^\pi \frac{R^2e^{2i\theta}iRe^{i\theta}\ d\theta}{R^4\left(e^{2i\theta}+\frac{1}{R^2}\right)\left(e^{2i\theta}+\frac{4}{R^2}\right)}$$

$$=\frac{i}{R}\int_0^\pi \frac{e^{3i\theta}\ d\theta}{\left(e^{2i\theta}+\frac{1}{R^2}\right)\left(e^{2i\theta}+\frac{4}{R^2}\right)} \to 0 \ \ (as\ R\to\infty)$$

hence $\quad I_1=\dfrac{\pi}{6}$

(ii)

$$I_2 = \int_0^\infty \frac{\cos x\, dx}{(x^2+1)^2} = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+1)^2}$$

$f(z) = \dfrac{e^{iz}}{(z^2+1)^2}$ has two poles of order two, $\quad z_0 = \pm i$

$$\int_C f(z)\, dz = 2\pi i\, b_1(i)$$

$$b_1(i) = \lim_{z\to i} \frac{d}{dz}\frac{e^{iz}}{[z+i]^2} = \left[\frac{i\, e^{iz}[z+i]^2 - 2e^{iz}[z+i]}{[z+i]^4}\right]_{z=i} = \frac{8}{16ei} = \frac{1}{2e\, i}$$

$$\int_C f(z)\, dz = 2\pi i\left(\frac{1}{2e\, i}\right) = \frac{\pi}{e} = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+1)^2} + \int_{C'} f(z)\, dz \quad (as\ R\to\infty)$$

$$\text{But } \int_{C'} f(z)\, dz = \int_0^\pi \frac{e^{iR\cos\theta - R\sin\theta} R e^{i\theta}\, i\, d\theta}{(R^2 e^{2i\theta}+1)^2}$$

$$= \frac{i}{R^3}\int_0^\pi \frac{e^{iR\cos\theta - R\sin\theta + i\theta} d\theta}{(e^{2i\theta} + \frac{1}{R^2})^2} \to 0 \quad (as\ R\to\infty)$$

Hence $\quad I_2 = \dfrac{\pi}{2e}$

(iii)

$$I_3 = \int_{-\pi}^{\pi} \frac{\cos\theta \, d\theta}{5+4\cos\theta} = \int_C \frac{(z^2+1)}{(2z)(\frac{2}{z})(z^2+\frac{5}{2}z+1)} \cdot \frac{dz}{iz}, \qquad C: |z| = 1$$

$$= \frac{1}{4i} \int_C \frac{(z^2+1)}{z(z^2+\frac{5}{2}z+1)} dz,$$

$$= \frac{1}{4i} \int_C \frac{z^2+1}{z(z+\frac{1}{2})(z+2)} dz,$$

$$= (2\pi i)\frac{1}{4i}\left\{\left[\frac{z^2+1}{(z+\frac{1}{2})(z+2)}\right]_{z=0} + \left[\frac{z^2+1}{z(z+2)}\right]_{z=-\frac{1}{2}}\right\}$$

$$= (2\pi i)\frac{1}{4i}\left\{1-\frac{5}{3}\right\} = \frac{\pi}{2}\left\{\frac{-2}{3}\right\} = \frac{-\pi}{3}$$

(iv)

$$I_4=\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2+4x+5}=\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{x^2+4x+5}$$

$f(z)=\dfrac{e^{iz}}{z^2+4z+5}$ has two poles of order one, $z_0=-2\pm i$

$$\int_C f(z)\,dz=2\pi i\,b_1(-2+i)$$

$$b_1(-2+i)=\lim_{z\to-2+i}\frac{e^{iz}}{[z+2+i]}=\frac{e^{-2i-1}}{2i}$$

$$\int_C f(z)\,dz=2\pi i\left(\frac{e^{-2i-1}}{2i}\right)=\pi e^{-2i-1}=\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{x^2+4x+5}+\int_{C'}f(z)\,dz\quad(as\ R\to\infty)$$

But $\displaystyle\int_{C'}f(z)\,dz=\int_{C'}\frac{e^{iz}}{z^2+4z+5}\,dz=\frac{1}{R}\int_0^{\pi}\frac{e^{iR\cos\theta-R\sin\theta+i\theta}\,i\,d\theta}{(1+\frac{4}{R}+\frac{5}{R^2})^2}\to 0\quad(as\ R\to\infty)$

Hence $\displaystyle I_4=\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{x^2+4x+5}=\frac{-\pi}{e}\sin 2$